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ALTERNATIVE FORMULATIONS TO OBTAIN THE EIGENSOLUTIONS OF A CONTINUOUS STRUCTURE TO WHICH SPRING–MASS SYSTEMS ARE ATTACHED

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1. INTRODUCTION

In structural dynamics, schemes such as the assumed-modes method [1] or the Lagrange multipliers formalism [2, 3] are often used to obtain the approximate modes of vibration of complex dynamical systems consisting of a continuous structure combined with various spring–mass attachments. The assumed-modes method [1] is a procedure for discretizing an arbitrary structure prior to obtaining the governing equations of motion. This method consists of assuming a solution of the free vibration problem in the form of a series composed of a linear combination of N spatial functions multiplied by the time-varying generalized co-ordinates. The spatial functions must satisfy the boundary conditions of the unconstrained system, defined here as the arbitrary structure without the constraints. This series is then substituted into the expressions for the kinetic and potential energies, thus reducing them to discrete form, and the equations of motion are derived by means of Lagrange's equations. Assuming R spring–mass systems are attached to the unconstrained system at distinct locations, then the mass and stiffness matrices of the combined system can be expressed as the sum of diagonal matrices and R rank-one matrices. The modes of vibration of the combined system correspond to the eigensolutions of an $N \times N$ generalized eigenvalue problem.

In reference [4], an approach to reduce the aforementioned generalized eigenvalue problem was presented. Specifically, for a system with R spring–mass attachments at distinct locations, the $N \times N$ generalized eigenvalue problem was manipulated so that its characteristic determinant is equivalent to that of a smaller $R \times R$ matrix (where it was assumed $R \ll N$, since in practice, a large number of component modes, N, is generally used to ensure convergence and sufficient accuracy), each element of which involves a sum of N terms.

Interestingly, this reduced characteristic determinant can also be obtained by using the Lagrange multipliers formalism [2, 3]. This method is based on using the spatial functions of the unconstrained structure in a Rayleigh–Ritz analysis with the constraint conditions enforced by means of Lagrange multipliers. Using this particular approach, R Lagrange multipliers and R constraint variables are introduced in the analysis. Manipulating the equations of motion, the eigenvalues must satisfy the zeros of the constraint equations in matrix form. Under certain conditions, the $R \times R$ characteristic determinant that needs to be solved is shown to be identical to that obtained by mathematically manipulating the $N \times N$ generalized eigenvalue problem as obtained from the assumed-modes method [4].

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Using the Lagrange multipliers formalism, Dowell [2] outlined the means to determine the eigenvectors of combined dynamical systems. However, he did not derive any expressions nor give any examples. In this technical note, the unconstrained modes of vibration will be used to extract the constrained modes of vibration of a continuous elastica to which spring–mass systems are attached. In real systems, having the attachment locations coincide exactly with the nodes of the unconstrained system is nearly impossible to achieve. Thus for simplicity, it will be assumed in this technical note that the attachment locations are distinct from the nodes of the unconstrained structure. Once the eigenvalues of the combined system are found by solving for the roots of a reduced characteristic determinant, a closed-form expression, derived from the general Lagrange multipliers formalism, will be provided that can be used to calculate the eigenvectors of the combined system.

2. THEORY

Consider the free vibration of the simple combined system of Figure 1, which consists of a uniform continuous system to which a grounded spring–mass system is attached at x_1. The modes of vibration of the system correspond to the eigensolutions of the generalized eigenvalue problem

$$[\mathscr{K}]\bar{\boldsymbol{\eta}} = \omega^2[\mathscr{M}]\bar{\boldsymbol{\eta}}, \tag{1}$$

where ω is the natural frequency of the combined system, and $\bar{\boldsymbol{\eta}}$ represents its corresponding eigenvector. The $N \times N$ stiffness and mass matrices, $[\mathscr{K}]$ and $[\mathscr{M}]$, are given by

$$[\mathscr{K}] = [\Lambda] + k\boldsymbol{\phi}_1\boldsymbol{\phi}_1^{\mathrm{T}}, \qquad [\mathscr{M}] = [I] + m\boldsymbol{\phi}_1\boldsymbol{\phi}_1^{\mathrm{T}}, \tag{2, 3}$$

where $[I]$ denotes the identity matrix, $[\Lambda]$ is a diagonal matrix whose ith element is given by λ_i, the square of the ith natural frequency of the unconstrained system, and $\boldsymbol{\phi}_1$ is a vector of the normalized eigenfunctions (such that the generalized masses are identically one) of the unconstrained system evaluated at x_1:

$$\boldsymbol{\phi}_1 = [\phi_1(x_1), \ldots, \phi_i(x_1), \ldots, \phi_N(x_1)]^{\mathrm{T}}. \tag{4}$$

Note that both $[\mathscr{K}]$ and $[\mathscr{M}]$ consist of a diagonal matrix modified by a rank-one matrix. For a non-trivial $\bar{\boldsymbol{\eta}}$, the eigenvalues, ω^2, must satisfy

$$\det([\mathscr{K}] - \omega^2[M]) = \det([\Lambda] + \sigma\boldsymbol{\phi}_1\boldsymbol{\phi}_1^{\mathrm{T}} - \omega^2[I]) = 0, \tag{5}$$

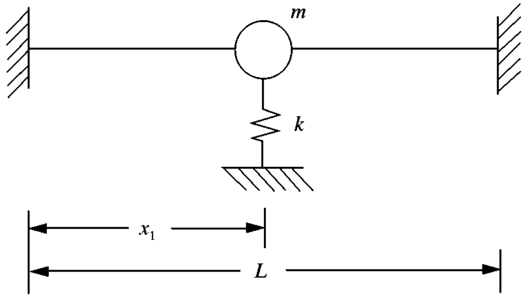

Figure 1. Combined dynamical system consisting of a continuous structure with a spring–mass system.

where $\sigma = k - m\omega^2$. Rearranging equation (5), it can be reduced to the following simple frequency equation (see Appendix A of reference [4] for detailed derivation):

$$\det([\Lambda] + \sigma\boldsymbol{\phi}_1\boldsymbol{\phi}_1^{\mathrm{T}} - \omega^2[I]) = \det([\Lambda] - \omega^2[I])\det([I] + \sigma([\Lambda] - \omega^2[I])^{-1}\boldsymbol{\phi}_1\boldsymbol{\phi}_1^{\mathrm{T}})$$

$$= \prod_{i=1}^{N}(\lambda_i - \omega^2)\left(1 + \sigma\sum_{i=1}^{N}\frac{\phi_i^2(x_1)}{\lambda_i - \omega^2}\right) = 0, \tag{6}$$

where $\phi_i(x_1)$ is the ith element of $\boldsymbol{\phi}_1$ at x_1. The eigenvalues of equation (1) also correspond to the zeros of equation (6), which can be determined either graphically or numerically by using any standard root solvers routine. Once the natural frequencies are found, the corresponding eigenvectors, the $\bar{\boldsymbol{\eta}}$'s, can be computed via Gaussian elimination by solving equation (1).

Alternatively, the method of Lagrange multipliers [2, 3] can also be used to analyze the free vibration problem of Figure 1. Using this particular approach (see reference [2] for detailed derivation), the equations of motion are given by

$$\ddot{\eta}_i + \lambda_i\eta_i - \mu\phi_i(x_1) = 0, \quad i = 1, \ldots, N, \tag{7}$$

$$kz + m\ddot{z} + \mu = 0, \tag{8}$$

where μ represents the Lagrange multiplier, and z is the spring–mass deflection. Assuming harmonic motion,

$$\eta_i(t) = \bar{\eta}_i\mathrm{e}^{\mathrm{j}\omega t}, \qquad \mu(t) = \bar{\mu}\mathrm{e}^{\mathrm{j}\omega t}, \qquad z(t) = \bar{z}\mathrm{e}^{\mathrm{j}\omega t} \tag{9}$$

and equations (7) and (8) become

$$(\lambda_i - \omega^2)\bar{\eta}_i - \bar{\mu}\phi_i(x_1) = 0, \quad i = 1, \ldots, N, \tag{10}$$

$$(k - \omega^2 m)\bar{z} + \bar{\mu} = 0. \tag{11}$$

Solving for $\bar{\eta}_i$ and $\bar{z}$ from equations (10) and (11), and substituting the resultant equations into the constraint equation

$$\sum_{i=1}^{N}\phi_i(x_1)\bar{\eta}_i - \bar{z} = 0. \tag{12}$$

the following secular equation is obtained:

$$1 + (k - m\omega^2)\sum_{i=1}^{N}\frac{\phi_i^2(x_1)}{\lambda_i - \omega^2} = 0. \tag{13}$$

Comparing equations (6) and (13), note the absence of the product terms. When the constraint location is not located at the node of any of the component modes, the eigenvalues of the constrained and unconstrained systems must be distinct; thus $\omega^2 \neq \lambda_i$, and equation (6) reduces to equation (13).

While laborious to apply, the method of Lagrange multipliers does conveniently lead to closed-form expressions for the eigenvectors of the system, which reveal the modal participation of the unconstrained modes. The elements of the eigenvector of equation (1)

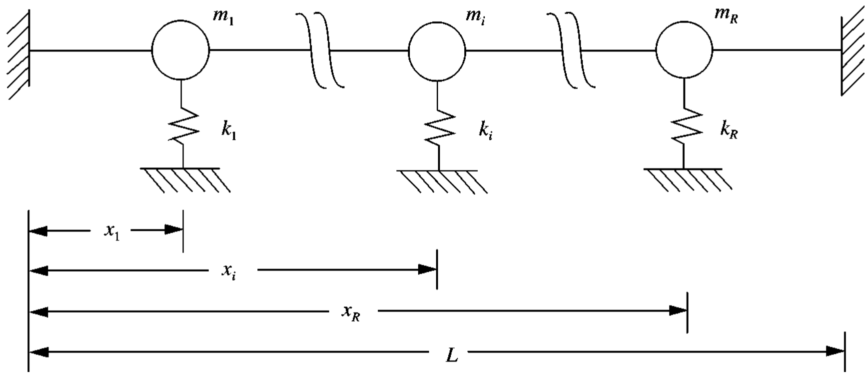

Figure 2. Combined dynamical system consisting of a continuous structure with R spring–mass systems.

are given by the $\bar{\eta}_i$'s. Thus, from equation (10), the ith element of the jth eigenvector of a continuous structure to which a spring–mass system is attached is

$$\bar{\eta}_i^j = \frac{\phi_i(x_1)}{\lambda_i - \omega_j^2}, \tag{14}$$

where the elements of the eigenvectors have been normalized by dividing them by $\bar{\mu}$.

The previous formalism is now extended to the more general case of a continuous structure to which R spring–mass systems are attached at distinct locations as shown in Figure 2. The modes of vibration of Figure 2 correspond to the eigensolutions of the generalized eigenvalue problem (1), where the stiffness and mass matrices are now given by (see, reference [4] for detailed derivation)

$$[\mathscr{K}] = [\Lambda] + \sum_{i=1}^{R} k_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^{\mathrm{T}}, \qquad [\mathscr{M}] = [I] + \sum_{i=1}^{R} m_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^{\mathrm{T}} \tag{15, 16}$$

and

$$\boldsymbol{\phi}_i = [\phi_1(x_i), \ldots, \phi_j(x_i), \ldots, \phi_N(x_i)]^{\mathrm{T}}. \tag{17}$$

Note that both $[\mathscr{K}]$ and $[\mathscr{M}]$ consist of a diagonal matrix modified by R rank one matrices. The natural frequencies of the combined system must satisfy the $N \times N$ characteristic determinant

$$\begin{aligned}\det([\mathscr{K}] - \omega^2[\mathscr{M}]) &= \det\left([\Lambda] + \sum_{i=1}^{R} k_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^{\mathrm{T}} - \omega^2 \sum_{i=1}^{R} m_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^{\mathrm{T}} - \omega^2 [I]\right)\\ &= \det\left([\Lambda] + \sum_{i=1}^{R} \sigma_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^{\mathrm{T}} - \omega^2 [I]\right) = 0,\end{aligned} \tag{18}$$

where $\sigma_i = k_i - m_i\omega^2$. Equation (18) can be expressed alternatively as the product of the following characteristic determinants:

$$\begin{aligned}\det\left([\Lambda] + \sum_{i=1}^{R} \sigma_i \boldsymbol{\phi}_i \boldsymbol{\phi}_i^{\mathrm{T}} - \omega^2[I]\right) &= \det([\Lambda] - \omega^2[I]) \det\left([I] + \sum_{i=1}^{R} \sigma_i ([\Lambda] - \omega^2[I])^{-1} \boldsymbol{\phi}_i \boldsymbol{\phi}_i^{\mathrm{T}}\right)\\ &= \prod_{i=1}^{N} (\lambda_i - \omega^2) \det[B] = 0,\end{aligned} \tag{19}$$

where the (i, j)th element of $[B]$, of size $R \times R$, is given by

$$b_{ij} = \sum_{r=1}^{N} \frac{\phi_r(x_i)\phi_r(x_j)}{\lambda_r - \omega^2} + \frac{1}{\sigma_i}\delta_i^j, \quad i, j = 1, \ldots, R. \tag{20}$$

Note that each element of $[B]$ consists of a sum of N terms. In equation (20), $\phi_r(x_i)$ denotes the rth eigenfunction at x_i and δ_i^j is the Kronecker delta. Once the natural frequencies, ω's, are found by solving for the roots of equation (19), the corresponding eigenvectors, $\bar{\boldsymbol{\eta}}$'s can be computed by solving equation (1) using Gaussian elimination, where the stiffness and mass matrices are now given by equations (15) and (16) respectively.

Like before, the method of Lagrange multipliers [2, 3] can also be used to analyze the free vibration problem of Figure 2. Applying Lagrange's equation yields

$$\ddot{\eta}_i + \lambda_i \eta_i - \sum_{r=1}^{R} \mu_r \phi_i(x_r) = 0, \quad i = 1, \ldots, N, \tag{21}$$

$$k_j z_j + m_j \ddot{z}_j + \mu_j = 0, \quad j = 1, \ldots, R, \tag{22}$$

where μ_j represents the jth Lagrange multiplier, and z_j is the spring–mass deflection at x_j. Assuming harmonic motion, equations (21) and (22) reduce to

$$(\lambda_i - \omega^2)\bar{\eta}_i - \sum_{r=1}^{R} \bar{\mu}_r \phi_i(x_r) = 0, \quad i = 1, \ldots, N, \tag{23}$$

$$k_j \bar{z}_j - \omega^2 m_j \bar{z}_j + \bar{\mu}_j = 0, \quad j = 1, \ldots, R. \tag{24}$$

Solving for the $\bar{\eta}_i$'s and $\bar{z}_i$'s in terms of the Lagrange multipliers in equations (23) and (24), and substituting the resultant equations into the constraint equations

$$\sum_{r=1}^{N} \phi_r(x_i)\bar{\eta}_r - \bar{z}_i = 0, \quad i = 1, \ldots, R, \tag{25}$$

R constraint equations are obtained that can be expressed in matrix form as

$$[B]\bar{\boldsymbol{\mu}} = \mathbf{0}, \tag{26}$$

where $\bar{\boldsymbol{\mu}} = [\bar{\mu}_1, \ldots, \bar{\mu}_R]^{\mathrm{T}}$, and $[B]$ is an order $R \times R$ matrix whose elements are given by equation (20). For non-trivial solution, the natural frequencies are given by the roots of the characteristic determinant

$$\det[B] = 0. \tag{27}$$

Comparing equations (27) and (19), note the absence of the product terms. When the constraint locations do not coincide with the nodes of the component modes, $\omega^2 \neq \lambda_i$, and equation (19) reduces to equation (27).

The eigenvectors of a continuous structure to which R spring–mass systems are attached can be readily obtained by using the Lagrange multipliers formalism. Using equation (23), the ith element of the jth eigenvector is

$$\bar{\eta}_i^j = \frac{1}{\lambda_i - \omega_j^2} \sum_{r=1}^{R} \bar{\mu}_r \phi_i(x_r), \tag{28}$$

where the Lagrange multipliers for the jth eigenvector, the $\bar{\mu}_r$'s, can be obtained by solving equation (26) using Gaussian elimination, for $\omega = \omega_j$. For $R = 1$, equation (14) is recovered if the eigenvectors are normalized by $\bar{\mu}_1$.

3. DISCUSSION AND RESULTS

The Lagrange multipliers formalism leads immediately to the much smaller characteristic determinant of order $R \times R$. This particular approach, however, requires R constraint variables, z_i's, and R Lagrange multipliers, μ_i's, to be introduced. For complicated systems, the task of obtaining the constraint equations may be lengthy and non-trivial, and this method loses much of its simplicity. While the Lagrange multipliers formalism is more tedious to apply, it does conveniently lead to closed-form expressions for the eigenvectors.

Based on the discussion above, an alternative formulation is proposed to analyze the free vibration of a combined dynamical system as follows.

(1) Use the simple assumed-modes method, in conjunction with Lagrange's equations, to formulate an $N \times N$ generalized eigenvalue problem.
(2) Manipulate the generalized eigenvalue problem to a smaller $R \times R$ characteristic determinant.
(3) Solve for the natural frequencies and evoke equation (28), obtained via the Lagrange multiplier formalism, to determine the corresponding eigenvectors.

This proposed scheme thus eliminates the need to solve an $N \times N$ generalized eigenvalue problem (which can be very costly), and it does not require one to introduce additional variables, the z_i's and μ_i's (which can be complicated). The utility of the proposed approach in determining the natural frequencies of combined systems was thoroughly investigated in reference [4], and the results were compared with those given in literature, obtained by using other methods. In this technical note, the proposed formulation will be used to obtain the eigenvalues and eigenvectors of a continuous structure to which a single spring–mass system and R spring–mass systems are attached.

3.1. SINGLE CONSTRAINT

For definiteness, the free vibration of a uniform fixed–fixed string under constant tension, T, with a spring–mass oscillator attached at x_1 as shown in Figure 1 will be analyzed. The component modes and the natural frequencies squared of the unconstrained system are given by the normalized eigenfunctions and the eigenvalues of a fixed–fixed spring, respectively,

$$\phi_i(x) = \sqrt{\frac{2}{\rho L}} \sin \frac{\mathrm{i}\pi x}{L}, \qquad \lambda_i = \left(\frac{\mathrm{i}\pi c}{L}\right)^2, \tag{29, 30}$$

where ρ represents the mass per unit length of the string, L the length of the string, and $c = \sqrt{T/\rho}$. The natural frequencies of the combined system can be calculated by solving for the eigenvalues of an $N \times N$ generalized problem (see equation (1)), or by solving for the roots of a secular equation (see equation (6)).

When $\phi_i(x_1) \neq 0$, for $i = 1, \ldots, N$ (physically, this occurs when the constraint location is not at any node of the first N component modes of the unconstrained system), the eigenvalues of the constrained system must be distinct from those of the unconstrained

TABLE 1

The first six natural frequencies of the system shown in Figure 1, for $k = 15T/L$, $m = 7\rho L$, $x_1 = 0{\cdot}78L$ and $N = 15$. The natural frequencies are non-dimensionalized by dividing by $\sqrt{T/\rho L^2}$

Natural frequency	Equation (1)	Equation (13)
1	1·6874	1·6874
2	4·1189	4·1189
3	8·1479	8·1479
4	12·1858	12·1858
5	14·7591	14·7591
6	16·3215	16·3215

TABLE 2

The first three normalized eigenvectors of the system shown in Figure 1, for $k = 15T/L$, $m = 7\rho L$, $x_1 = 0{\cdot}78L$ and $N = 15$, where $\bar{\boldsymbol{\eta}}_A^j$ = jth eigenvector obtained from solving equation (1), and $\bar{\boldsymbol{\eta}}_B^j$ = jth eigenvector obtained from solving equation (14)

$\bar{\boldsymbol{\eta}}_A^1$	$\bar{\boldsymbol{\eta}}_B^1$	$\bar{\boldsymbol{\eta}}_A^2$	$\bar{\boldsymbol{\eta}}_B^2$	$\bar{\boldsymbol{\eta}}_A^3$	$\bar{\boldsymbol{\eta}}_B^3$
−1·000E+00	−1·000E+00	−1·000E+00	−1·000E+00	−2·888E−01	−2·888E−01
2·954E−01	2·954E−01	−4·857E−01	−4·857E−01	9·347E−01	9·347E−01
−1·123E−01	−1·123E−01	1·358E−01	1·358E−01	1·000E+00	1·000E+00
2·615E−02	2·615E−02	−2·907E−02	−2·907E−02	−1·030E−01	−1·030E−01
1·396E−02	1·396E−02	−1·497E−02	−1·497E−02	−4·387E−02	−4·387E−02
−2·639E−02	−2·639E−02	2·778E−02	2·778E−02	7·483E−02	7·483E−02
2·273E−02	2·273E−02	−2·367E−02	−2·367E−02	−6·089E−02	−6·089E−02
−1·199E−02	−1·199E−02	1·240E−02	1·240E−02	3·101E−02	3·101E−02
8·684E−04	8·684E−04	−8·933E−04	−8·933E−04	−2·193E−03	−2·193E−03
6·580E−03	6·580E−03	−6·746E−03	−6·746E−03	−1·635E−02	−1·635E−02
−8·956E−03	−8·956E−03	9·159E−03	9·159E−03	2·199E−02	2·199E−02
7·028E−03	7·028E−03	−7·173E−03	−7·173E−03	−1·710E−02	−1·710E−02
−2·817E−03	−2·817E−03	2·871E−03	2·871E−03	6·807E−03	6·807E−03
−1·418E−03	−1·418E−03	1·444E−03	1·444E−03	3·409E−03	3·409E−03
4·019E−03	4·019E−03	−4·087E−03	−4·087E−03	−9·616E−03	−9·616E−03

system, i.e., $\omega^2 \neq \lambda_i$. Thus, the eigenvalues of the constrained system can be obtained by solving for the roots of equation (13). Table 1 lists the first six natural frequencies of the system of Figure 1, for $k = 15T/L$, $m = 7\rho L$, $N = 15$ and $x_1 = 0{\cdot}78L$, where the constraint location does not coincide with the node of any of the first 15 modes. Note that equations (1) and (13) give identical results. The eigenvectors of the system can be obtained either by solving the generalized eigenvalue problem of equation (1), or by using equation (14), which leads immediately to closed-form expressions once the natural frequencies are known. Table 2 shows the first three eigenvectors of the system, where the elements of the eigenvectors are normalized with respect to the magnitude of the largest entry. Note again the exact agreement between the two schemes.

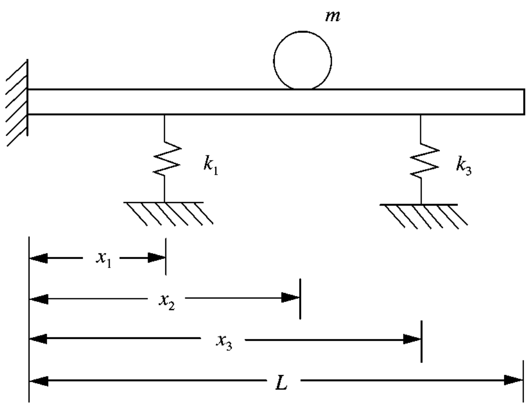

Figure 3. Uniform cantilever Euler–Bernoulli beam with three lumped attachments.

3.2. MULTIPLE CONSTRAINTS

Here the free vibration of a uniform cantilever Euler–Bernoulli beam to which two grounded translational springs and one lumped mass are attached as shown in Figure 3 will be analyzed. The component modes and the natural frequencies squared of the unconstrained system correspond to the normalized eigenfunctions and the eigenvalues of a uniform fixed–free Euler–Bernoulli beam, respectively,

$$\phi_i(x) = \frac{1}{\sqrt{\rho L}}\left(\cos\beta_i x - \cosh\beta_i x + \frac{\sin\beta_i L - \sinh\beta_i L}{\cos\beta_i L + \cosh\beta_i L}(\sin\beta_i x - \sinh\beta_i x)\right), \quad (31)$$

$$\lambda_i = (\beta_i L)^4, \quad (32)$$

where ρ represents the mass per unit length of the beam, L its length, and $\beta_i L$ satisfies the following transcendental equation:

$$\cos\beta_i L \cosh\beta_i L = -1. \quad (33)$$

When the constraints are not at the nodes of the component modes, then $\omega^2 \neq \lambda_i$ and the eigenvalues are given by the roots of equation (27). Table 3 lists the first six natural frequencies of the combined beam system of Figure 3, for $(k_1, m_1) = (12EI/L^3, 0)$, $(k_2, m_2) = (0, 2\rho L)$, $(k_3, m_3) = (10EI/L^3, 0)$, $x_1 = 0{\cdot}30L$, $x_2 = 0{\cdot}60L$, $x_3 = 0{\cdot}80L$, and $N = 15$. The natural frequencies are non-dimensionalized by dividing by $\sqrt{EI/\rho L^4}$. Note that both equations (1) and (27) give identical results.

The eigenvectors of Figure 3 can be obtained by using either equation (1) or equation (28). The former requires one to solve an $N \times N$ generalized eigenvalue problem, while the latter leads immediately to closed-form expressions once the natural frequencies are known and the Lagrange multipliers, $\bar{\mu}_r$'s, are determined. Theoretically, the $\bar{\mu}_r$'s can be computed via Gaussian elimination by solving equation (26). However, because this technique is purely numerical, the eigenvalues thus obtained are close but not quite equal to the exact eigenvalues. Thus, matrix $[B]$ of equation (26) is never singular, which leads to trivial Lagrange multipliers, $\bar{\boldsymbol{\mu}} = \mathbf{0}$, and trivial eigenvectors, $\bar{\boldsymbol{\eta}} = \mathbf{0}$. To circumvent this numerical difficulty, $\bar{\mu}_1$ is arbitrarily set to 1, since the mode shapes are unique up to an arbitrary constant. Rearranging equation (26) leads to

$$[B']\bar{\boldsymbol{\mu}}' = \mathbf{f}. \quad (34)$$

TABLE 3

The first six natural frequencies of the system shown in Figure 3, for $(k_1, m_1) = (12EI/L^3, 0)$, $(k_2, m_2) = (0, 2\rho L)$, $(k_3, m_3) = (10EI/L^3, 0)$, $x_1 = 0{\cdot}30L$, $x_2 = 0{\cdot}60L$, $x_3 = 0{\cdot}80L$, *and* $N = 15$. *The natural frequencies are non-dimensionalized by dividing by* $\sqrt{EI/\rho L^4}$

Natural frequency	Equation (1)	Equation (27)
1	3·4785	3·4785
2	15·7448	15·7448
3	53·8761	53·8761
4	113·0797	113·0797
5	164·8772	164·8772
6	297·2032	297·2032

TABLE 4

The first three normalized eigenvectors of the system shown in Figure 3, for $(k_1, m_1) = (12EI/L^3, 0)$, $(k_2, m_2) = (0, 2\rho L)$, $(k_3, m_3) = (10EI/L^3, 0)$, $x_1 = 0{\cdot}30L$, $x_2 = 0{\cdot}60L$, $x_3 = 0{\cdot}80L$, *and* $N = 15$, *where* $\bar{\boldsymbol{\eta}}_A^j$ = *jth eigenvector obtained from solving equation* (1), *and* $\bar{\boldsymbol{\eta}}_B^j$ = *jth eigenvector obtained from solving equation* (28)

$\bar{\boldsymbol{\eta}}_A^1$	$\bar{\boldsymbol{\eta}}_B^1$	$\bar{\boldsymbol{\eta}}_A^2$	$\bar{\boldsymbol{\eta}}_B^2$	$\bar{\boldsymbol{\eta}}_A^3$	$\bar{\boldsymbol{\eta}}_B^3$
−1·000E+00	−1·000E+00	9·030E−01	9·030E−01	3·195E−01	3·195E−01
−5·571E−02	−5·571E−02	−1·000E+00	−1·000E+00	4·983E−01	4·983E−01
4·561E−03	4·561E−03	6·285E−02	6·285E−02	1·000E+00	1·000E+00
2·577E−03	2·577E−03	8·815E−03	8·815E−03	5·400E−02	5·400E−02
−1·320E−03	−1·320E−03	−7·096E−03	−7·096E−03	−3·752E−02	−3·752E−02
1·008E−04	1·008E−04	3·023E−04	3·023E−04	2·996E−03	2·996E−03
1·690E−04	1·690E−04	1·454E−03	1·454E−03	7·531E−03	7·531E−03
−1·254E−04	−1·254E−04	−6·377E−04	−6·377E−04	−3·278E−03	−3·278E−03
1·800E−05	1·800E−05	−2·865E−04	−2·865E−04	−1·319E−03	−1·319E−03
2·606E−05	2·606E−05	4·081E−04	4·081E−04	1·715E−03	1·715E−03
4·756E−06	4·756E−06	−4·400E−05	−4·400E−05	−1·951E−04	−1·951E−04
−1·785E−05	−1·785E−05	−1·619E−04	−1·619E−04	−7·262E−04	−7·262E−04
1·777E−06	1·777E−06	8·960E−05	8·960E−05	4·345E−04	4·345E−04
1·006E−05	1·006E−05	3·349E−05	3·349E−05	2·061E−04	2·061E−04
−1·245E−05	−1·245E−05	−6·257E−05	−6·257E−05	−3·276E−04	−3·276E−04

In equation (34), $[B']$ is the lower $(R-1)\times(R-1)$ matrix of $[B]$, $\bar{\boldsymbol{\mu}}' = [\bar{\mu}_2, \ldots, \bar{\mu}_R]^{\mathrm{T}}$, and $\mathbf{f} = [-b_{21}, \ldots, -b_{i1}, \ldots, -b_{R1}]^{\mathrm{T}}$, where b_{i1} denotes the $(i, 1)$th element of matrix $[B]$ (see equation (20) for the expressions of b_{i1}). Solving for $\bar{\boldsymbol{\mu}}'$ in equation (34), non-trivial solutions for the Lagrange multipliers are obtained. Applying equation (28), the corresponding eigenvectors can be readily calculated. Table 4 shows the first three eigenvectors of the system of Figure 3, where the elements of the eigenvectors are normalized such that the magnitude of the largest entry becomes 1. Note the exact agreement between the results.

4. CONCLUSIONS

An alternative formalism that combines the best features of the assumed-mode method and the Lagrange multipliers formalism is introduced to analyze the free vibration of a continuous structure with spring–mass attachments. Using the classical assumed-modes method in conjunction with Lagrange's equations, the natural frequencies are obtained by solving the roots of an $N \times N$ characteristic determinant. Algebraically manipulating this characteristic determinant, it can be reduced to a smaller one of size $R \times R$, the same solution that is obtained by applying the more laborious Lagrange multipliers formalism. Once the eigenvalues have been calculated, closed-form expressions, obtained with the Lagrange multipliers formalism, can be used to determine the eigenvectors of the combined system without solving the generalized eigenvalue problem associated with the combined system.

REFERENCES

1. L. MEIROVITCH 1967 *Analytical Methods in Vibrations*. New York: The Macmillan Company.
2. E. H. DOWELL 1971 *Journal of Applied Mechanics* **38**, 595–600. Free vibrations of a linear structure with arbitrary support conditions.
3. E. H. DOWELL 1972 *Journal of Applied Mechanics* **39**, 727–732. Free vibrations of an arbitrary structure in terms of component modes.
4. P. D. CHA and W C. WONG 1999 *Journal of Sound and Vibration* **219**, 689–706. A novel approach to determine the frequency equations of combined dynamical systems.